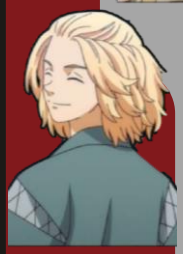
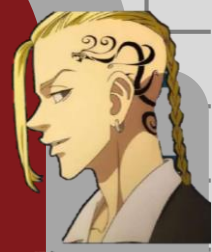
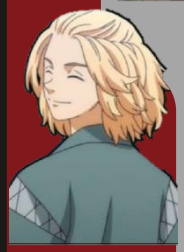


Chapter 2: Matrices

by : rraezzz

❑ Matrices is a set of numbers arranged in rows and columns to form a rectangular or a square array





ition:

row 1 → $[1700 \quad 2100 \quad 2000 \quad 1800]$ or $\begin{bmatrix} 1700 \\ 2100 \\ 2000 \\ 1800 \end{bmatrix}$

This is a row matrix. A row matrix has only one row.

column 1

This is a column matrix. A column matrix has only one column.

row 1 → row 2 → row 3 → $\begin{bmatrix} 13 & 5 & 1 \\ 10 & 2 & 4 \\ 8 & 8 & 9 \end{bmatrix}$ or $\begin{bmatrix} 13 & 10 & 8 \\ 5 & 2 & 8 \\ 1 & 4 & 9 \end{bmatrix}$

column 1 column 2 column 3

These are square matrices. A square matrix has the same number of rows and columns.

row 1 → row 2 → row 3 → $\begin{bmatrix} 76 & 80 \\ 82 & 88 \\ 72 & 70 \end{bmatrix}$ or row 1 → row 2 → $\begin{bmatrix} 76 & 82 & 72 \\ 80 & 88 & 70 \end{bmatrix}$

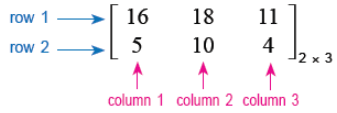
column 1 column 2

column 1 column 2 column 3

These are rectangular matrices. A rectangular matrix has different number of rows and columns.

by : rraezzz

❑ Order of a matrix can be determined by counting the number of rows followed by the number of columns of the matrix



This matrix has 2 rows and 3 columns. Therefore, it is a matrix with order 2 × 3 and can be read as "matrix 2 by 3".

Matrix with m rows and n columns has the order $m \times n$ and is read as "matrix m by n".

Every number in the matrix is known as an **element** of the matrix. For example, the element at the 2nd row and 3rd column for the matrix $\begin{bmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix}$ is 4.

Capital letter is used to denote a matrix, for example $A = \begin{bmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix}$, and the element at the 2nd row and 3rd column can be represented as a_{23}, for example $a_{23} = 4$.

a_{ij}

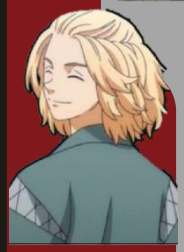
i^{th} row

j^{th} column

Hence, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

by : rraezzz

- M equal to N if andonly if both the matrices have the same order and each corresponding element is equal



Determine whether each of the following pairs of matrices is equal. Give your reason.

(a) $A = \begin{bmatrix} 2 & 11 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 11 \\ 1 & 3 \end{bmatrix}$

(b) $C = [3 \quad 9]$ and $D = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

(c) $E = \begin{bmatrix} 8 & 3 \\ -7 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 8 & -7 \\ 3 & 0 \end{bmatrix}$

(d) $G = \begin{bmatrix} 0.5 & -2 \\ 6 & 0.8 \\ -1 & 12 \end{bmatrix}$ and $H = \begin{bmatrix} \frac{1}{2} & -2 \\ 6 & \frac{4}{5} \\ -1 & 12 \end{bmatrix}$

Solution:

(a) $A = B$ because both the matrices have the same order and each corresponding element is equal.

(b) $C \neq D$ because both the matrices do not have the same order. The order of C is 1×2 while the order of D is 2×1.

(c) $E \neq F$ because the corresponding elements are different.

(d) $G = H$ because both the matrices have the same order and each corresponding element is equal.

It is given that matrix $P = \begin{bmatrix} x & 7 \\ 0 & 5 - 3z \end{bmatrix}$ and matrix $Q = \begin{bmatrix} 5 & y + 1 \\ 0 & 2z \end{bmatrix}$. Determine the values of x, y and z if $P = Q$.

Solution:

$P = Q$, hence all the corresponding elements are equal.

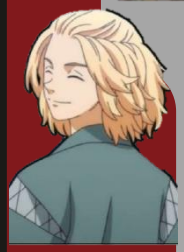
$x = 5$, $7 = y + 1$, $5 - 3z = 2z$

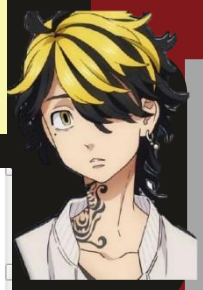
$y = 7 - 1$ $5 = 5z$

$y = 6$ $z = 1$

by : rraezzz

- Addition and subtraction of matrices can only be performed on the matrices with the same order
- Each corresponding element is added or subtracted to obtain a single matrix with the same order





For matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and matrix $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \text{ and } A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}.$$

Example 7

It is given that matrix $C = \begin{bmatrix} 10 & -8 & 4 \\ 6 & -11 & 7 \end{bmatrix}$, matrix $D = \begin{bmatrix} 14 & -2 & 1 \\ -3 & 5 & 9 \end{bmatrix}$, matrix $P = \begin{bmatrix} -2 & \frac{5}{6} & 6 \\ 7.4 & -13 & 5 \\ 1 & 9 & \frac{1}{4} \end{bmatrix}$ and matrix $Q = \begin{bmatrix} 18 & \frac{1}{3} & -7 \\ 2.5 & -8 & 3 \\ 12 & 0 & 0.4 \end{bmatrix}$. Calculate

(a) $C + D$ (b) $P - Q$

Solution:

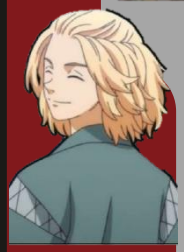
(a) $C + D = \begin{bmatrix} 10 & -8 & 4 \\ 6 & -11 & 7 \end{bmatrix} + \begin{bmatrix} 14 & -2 & 1 \\ -3 & 5 & 9 \end{bmatrix}$

$= \begin{bmatrix} 10 + 14 & -8 + (-2) & 4 + 1 \\ 6 + (-3) & -11 + 5 & 7 + 9 \end{bmatrix}$ ← Add corresponding elements

$= \begin{bmatrix} 24 & -10 & 5 \\ 3 & -6 & 16 \end{bmatrix}$

by : rraezzz

❑ Multiplication of a matrix by a number is known as scalar multiplication



Given that $D = \begin{bmatrix} -5 & 4 \\ 2 & 1 \end{bmatrix}$, calculate

(a) $3D$

(b) $-\frac{1}{2}D$

Solution:

(a) $3D = 3\begin{bmatrix} -5 & 4 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3(-5) & 3(4) \\ 3(2) & 3(1) \end{bmatrix}$ ← Multiply all elements by 3

$= \begin{bmatrix} -15 & 12 \\ 6 & 3 \end{bmatrix}$

(b) $-\frac{1}{2}D = -\frac{1}{2}\begin{bmatrix} -5 & 4 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} \left(-\frac{1}{2}\right)(-5) & \left(-\frac{1}{2}\right)(4) \\ \left(-\frac{1}{2}\right)(2) & \left(-\frac{1}{2}\right)(1) \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{2} & -2 \\ -1 & -\frac{1}{2} \end{bmatrix}$

The results of Mind Mobilisation 1 show that;

(a) $A + B = B + A$. Addition of matrices obeys the **Commutative Law**.
$A - B \neq B - A$. Subtraction of matrices does not obey the Commutative Law.

(b) $h(A + B) = hA + hB$, $h(A - B) = hA - hB$.
Addition and subtraction of matrices obey the **Distributive Law**.

(c) $(A + B) + C = A + (B + C)$. Addition of matrices obeys the **Associative Law**.
$(A - B) - C \neq A - (B - C)$. Subtraction of matrices does not obey the Associative Law.

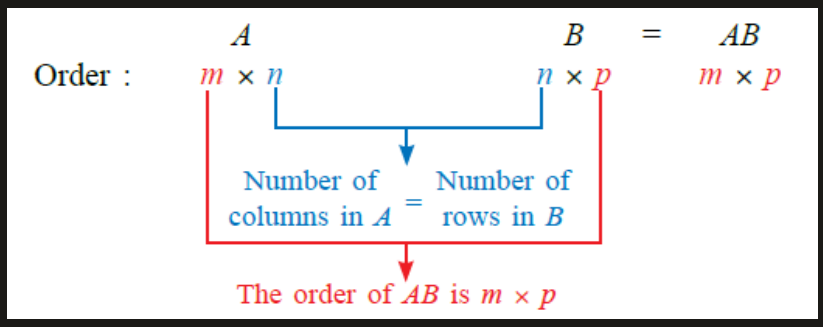
(d) A matrix with all its elements equal to zero is called **zero matrix**, for example $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Addition and subtraction of matrix A with zero matrix, O are:
$A + O = A$ and $A - O = A$

Info Bulletin

Example of zero matrix:
$O_{1\times 2} = [0 \quad 0]$
$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

by : rraezzz

- ❑ Multiply two matrices
- ❑ If matrix A has an order of m x n and matrix B has an order of n x p, then the multiplication AB can be performed and the order of AB is m x p



It is given that matrix $C = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ and matrix $D = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}$. Determine whether multiplication CD and DC can be performed. If yes, state the order of the product of the matrices.

Solution:

Order of matrix $C = 2 \times 1$, order of matrix $D = 2 \times 2$

C D
2×1 2×2
Not the same

Multiplication CD cannot be performed because the number of columns in matrix C is not the same with the number of rows in matrix D.

D C
2×2 2×1
Same
Order of DC

Multiplication DC can be performed because the number of columns in matrix D is the same with the number of rows in matrix C. The order of DC is 2×1.

by : rraezzz

❑ Identity matrix

Matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is known as identity matrix and is denoted by I.

Identity matrices are square matrices. **Identity matrix**, I, when multiplied with a matrix A, will result in matrix A.

$$AI = IA = A$$

Write the identity matrix based on the order given below.

(a) 1×1 (b) 2×2

(c) 4×4 (d) 5×5

Solution:

(a) $[1]$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

by : rraezzz

❑ Inverse Matrix

Determine whether the following matrix is the inverse matrix of $\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$. Explain your answer.

(a) $\begin{bmatrix} 4 & 1 \\ -7 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

Solution:

(a) $\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 14 & 11 \end{bmatrix}$ ← The product is not an identity matrix

$\begin{bmatrix} 4 & 1 \\ -7 & 2 \end{bmatrix}$ is not the inverse matrix of $\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ because the product of these two matrices is not an identity matrix.

(b) $\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← The product is an identity matrix

$\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← The product is an identity matrix

$\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ is the inverse matrix of $\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ because the product of these two matrices is an identity matrix.

For each of the following matrices, determine whether the inverse matrix exists. If yes, calculate the inverse matrix.

(a) $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

Solution:

(a) $ad - bc = 1(8) - 2(4)$ ← Determine the existence of inverse matrix with matrix determinant, $ad - bc$
$= 8 - 8$
$= 0$

$|A| = 0$. Hence, A^{-1} does not exist.

(b) $ad - bc = 3(4) - 5(2)$
$= 12 - 10$
$= 2$
$\neq 0$

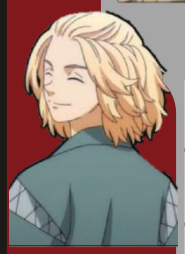
$|B| \neq 0$. Hence, B^{-1} exists.

$$B^{-1} = \frac{1}{3(4) - 5(2)}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$$

← Exchange the position of the elements in the main diagonal and multiply the other two elements with −1



by : rraezzz

❑ Method to solve simultaneous linear equations

Simultaneous linear equations

$ax + by = p$

$cx + dy = q$

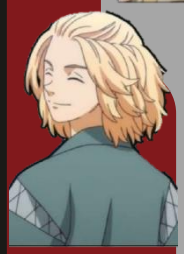
Matrix form $AX = B$

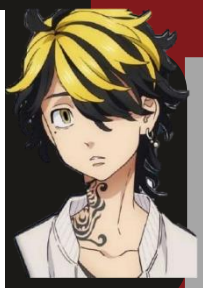
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

where a, b, c, d, p and q are constants while x and y are unknowns

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$





Solve the simultaneous linear equations below using the matrix method.

$$x - 2y = 1$$
$$3x - 4y = 4$$

Solution:

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$ ← Write the simultaneous linear equations in the form of matrix

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(1)(-4) - (-2)(3)}\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$$

Hence, $x = 2$ and $y = \frac{1}{2}$ ← Final answer

Smart TiPS

Solving simultaneous linear equations means calculating the values of x and y. Therefore, the final answer states the values.

Checking Answer

Multiply $\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 - 1 \\ 6 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

by : rraezzz